Dynamics and thermodynamics of a probe brane in the multicenter and rotating D3-brane background

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Abstract

We study the dynamics and thermodynamics of a probe D3-brane in the rotating D3-brane background and in its extremal limit, which is a multicenter configuration of D3-branes distributed uniformly on a disc. In the extremal background, if the angular momentum of the probe does not vanish, the probe is always bounced back at some turning point. When its angular momentum vanishes, in the disc plane, the probe will be captured at the edge of the disc; in the hyperplane orthogonal to the disc, the probe will be absorbed at the center of the disc. In the non-extremal background, if the probe is in the hyperplane orthogonal to the disc, it will be captured at the horizon; if the probe is restricted in the disc plane, the probe will be bounced back at a turning point, which is just the infinite red-shift hyperplane of the rotating background, even when the angular momentum of the probe vanishes. The thermodynamics of a relative static D3-brane probe is also investigated to the rotating D3-brane source. Two critical points are found. One is just the thermodynamically stable boundary of the source rotating D3-branes; the other is related to the distance between the probe and the source, which can be regarded as the mass scale in the corresponding super Yang-Mills theory. If the probe is static, the second critical point occurs as the probe is at the infinite red-shift hyperplane of the background.

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I. INTRODUCTION

Over the past several years the probe method has been used extensively in investigating the structure of black holes, the bound state of branes, dynamics and statistical mechanics of branes, and so on [1–10]. It turns out that many calculations involving the probe in supergravity backgrounds are in agreement with those obtained from the point of view of field theory.

On the other hand, there are two branches in the $\mathcal{N}$=4 super Yang-Mills (SYM) theory in four dimensions. The Higgs branch corresponds to the vevs of scalar fields being zero, while the gauge theory is in the Coulomb branch when the vevs of some scalar fields do not vanish. According to the Maldacena's conjecture [11], different states in the SYM theory can be described in terms of different configurations in supergravity. For instance, a vacuum state of $\mathcal{N}$=4 SYM theory with gauge group $U(N)$ in the Higgs branch is described by N extremal D3-branes coinciding with each other, and a thermal state of the theory is supposed to be described by the near-extremal D3-branes. Furthermore, the states of the SYM in the Coulomb branch are described by multicenter D3-brane solutions. A simplest case is that N parallel coinciding D3-branes are separated along a single transverse direction by a distance from a single D3-brane. In this case, the gauge symmetry $U(N+1)$ of the gauge theory is broken to $U(N)\times U(1)$.

However, unlike the single-center configurations, the multicenter solutions have not the non-extremal generalizations. This results in the difficulty to investigate thermodynamics of the SYM in Higgs phase. Recently Tseytlin and Yankielowicz [12] attacked this issue and studied the free energy of the SYM in the Higgs phase by using the probe method. They interpreted the supergravity interaction potential between near-extremal D3-branes (as source) and a D3-brane (as a probe) as contribution of massive states to the free energy of the large N SYM theory at strong 't Hooft coupling. In this method, it is worth noting that the source is excited, that is, the source is near-extremal static D3-branes, while the probe does not get excited. In this way some interesting results were observed. For example, from the free energy of the probe they predicted the existence of a phase transition. Note that there is no phase transition in the conformal case [13]. Therefore phase transition may be present in the Higgs phase. In the low temperature limit they found that the structure of terms which appear in the free energy at strong and weak couplings is same.

Recently, it has been found that it is possible to have the non-extremal generalizations of multicenter D3-brane configurations. But these D3-branes are distributed continuously, rather than discretely. The non-extremal generalizations of the continuously distributed D3-branes are just the rotating D3-brane solution found in [14–17], based on [18]. It means that the SYM corresponding to the rotating D3-branes is in the Higgs phase. Indeed, from the calculations in supergravity, it is learned that the behavior of the SYM at strong 't Hooft coupling is quite different in the different branches. For example, in the Higgs branch the quark-antiquark potential has the Coulombic behavior [19,20], while in the multicenter backgrounds the potential has not only the Coulombic behavior, but also the confining behavior [21–23].Their thermodynamic behaviors are also quite different. The thermodynamics of the static D3-branes is stable; the heat capacity is always positive-definite and hence the SYM is always in the unconfinement phase [13]. Furthermore the so-called localization instability for the static D3-brane configurations does not happen [24]. For the rotating D3-branes, however, it has been found that the thermodynamics of excitations of D3-branes is stable up to a critical angular momentum density, beyond which the heat capacity will become negative, and a phase transition may happen [25–27]. The localization instability may occur when the angular momentum reaches some critical value [28].

In this paper, we would like to investigate the structure and thermodynamics of the multicenter and rotating D3-brane configurations by using the probe method. In the next section we analyze the dynamics of a D3-brane probe in this multicenter and rotating D3-brane background. The thermodynamics of the probe will be investigated in section III. A comparison to the case of single-center solution is also made.

II. DYNAMICS OF A PROBE D3-BRANE IN THE MULTICENTER AND ROTATING D3-BRANE BACKGROUND

The D3-brane solutions in the type IIB supergravity have six transverse spatial dimensions. Therefore the rotating D3-brane solutions may have three independent angular momentum parameters. For simplicity, we consider the rotating D3-brane solution with only an angular momentum parameter. In this case, its metric is

$$ds^2 = \frac{1}{\sqrt{f}}\left(-hdt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \sqrt{f}\left[\frac{dr^2}{\tilde{h}} - \frac{4ml\cosh\alpha}{r^4\triangle f}\sin^2\theta dt d\phi \right.$$
$$\left. + r^2(\triangle d\theta^2 + \tilde{\triangle}\sin^2\theta d\phi^2 + \cos^2\theta d\Omega_3^2)\right], \tag{1}$$

where

$$f = 1 + \frac{2m\sinh^2\alpha}{r^4\triangle}, \quad \triangle = 1 + \frac{l^2\cos^2\theta}{r^2}, \quad \tilde{\triangle} = 1 + \frac{l^2}{r^2} + \frac{2ml^2\sin^2\theta}{r^6\triangle f},$$
$$h = 1 - \frac{2m}{r^4\triangle}, \quad \tilde{h} = \frac{1}{\triangle}\left(1 + \frac{l^2}{r^2} - \frac{2m}{r^4}\right). \tag{2}$$

In this solution the dilaton is a constant and the nonvanishing components of the four-form potential are

$$C = -\frac{(f^{-1}-1)}{\sinh\alpha}dx_1\wedge dx_2\wedge dx_3\wedge(\cosh\alpha dt - l\sin^2\theta d\phi). \tag{3}$$

Corresponding to this rotating D3-brane solution, the R-symmetry of SYM theory is broken from $SO(6)$ to $SO(4)\times U(1)$. Furthermore, it is observed that the extremal limit of this solution, $m\to 0$ keeping l fixed and $me^{2\alpha}$ finite, is a multicenter solution, which represents D3-branes to be distributed uniformly on a disc with a radius l [15,16]. That is, this extremal limit is some superposition of N static D3-branes, rather than that of N coinciding rotating D3-branes. Taking an appropriate coordinate transformation, this limiting metric becomes

$$ds^2 = H_0^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + H_0^{1/2}(dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2 + dy_5^2 + dy_6^2), \tag{4}$$

where

$$H_0 = 1 + \frac{2R^4}{[r^2 - l^2 + \sqrt{(r^2+l^2)^2 - 4l^2\rho^2}]\sqrt{(r^2+l^2)^2 - 4l^2\rho^2}}. \tag{5}$$

Here $r^2 = y_1^2 + \cdots + y_6^2$, $\rho^2 = y_5^2 + y_6^2$, and $R^4 = 4\pi g_s N\alpha'^2$. The four-form potential reduces to

$$C = (H_0^{-1} - 1)dt \wedge dx_1 \wedge dx_2 \wedge dx_3. \tag{6}$$

On the other hand, the dynamics of a D3-brane probe is governed by the following action:

$$S = -T_3 \int d^4x \sqrt{-\det \hat{G}} - T_3 \int \hat{C}, \tag{7}$$

where $T_3 = 1/2\pi g_s(2\pi\alpha')^2 = N/2\pi^2 R^4$ is the D3-brane tension. In order to investigate the dynamics of the probe, it is convenient to take the static gauge: $\tau = t$, x_i act as just the worldvolume coordinates, and other transverse coordinates y_i depend on τ only. Let's first discuss the case of the probe in the extremal background (4).

(i). *In the extremal background.* Substituting (4) and (6) into (7), we obtain

$$S = -T_3 V_3 \int d\tau H_0^{-1}[\sqrt{1 - H_0 \dot{y}_i^2} - 1], \tag{8}$$

where the overdot denotes derivative with respect to τ, V_3 is the spatial volume of the worldvolume and we subtracted a constant term. For a static probe, it is obvious from (8) that the interaction potential vanishes, which means that the source is indeed a BPS configuration. For a low-velocity probe, its action is, up to the term $\mathcal{O}(v^4)$,

$$S = \frac{T_3 V_3}{2} \int d\tau \dot{y}_i^2 + \mathcal{O}(v^4). \tag{9}$$

It is seen that the motion of the probe is like the motion of a test particle with mass $m_p = T_3 V_3$ in a flat spacetime. Therefore, to this order, its motion is the same as that in a single-center D-brane background [2]. Of course, this is also required by the BPS property of the system consisting of the source and the probe.

Now we consider a general motion of the probe in the extremal background. Generally speaking, the motion of the probe is like that of a test particle with mass m_p moving in a velocity-dependent potential. Set the probe have the angular momentum only in some ϕ-direction, we can then write down: $\dot{y}_i^2 = \dot{r}^2 + r^2\dot{\phi}^2$. From (8) we have the angular momentum L of the probe,

$$L = \frac{m_p r^2 \dot{\phi}}{\sqrt{1 - H_0(\dot{r}^2 + r^2\dot{\phi}^2)}}. \tag{10}$$

And the energy E of the probe is

$$E = \frac{m_p(\dot{r}^2 + r^2\dot{\phi}^2)}{\sqrt{1 - H_0(\dot{r}^2 + r^2\dot{\phi}^2)}} + m_p H_0^{-1}[\sqrt{1 - H_0(\dot{r}^2 + r^2\dot{\phi}^2)} - 1]. \tag{11}$$

From the above equation, using the expression of the angular momentum and the kinetic relation,

$$E = \frac{1}{2} m_p \dot{r}^2 + V(r), \tag{12}$$

we can obtain an effective potential of the radial motion of the probe,

$$V(r) = E\left[1 - \frac{1 + EH_0/2m_p}{(1 + EH_0/m_p)^2}\right] + \frac{L^2}{2m_p r^2}\frac{1}{(1 + EH_0/m_p)^2}. \tag{13}$$

Thus, the radial motion of the probe is that of a test particle with mass m_p moving in the velocity-independent central force potential $V(r)$. Characterizing the motion is the turning points, which satisfy the equation $E = V(r_c)$. It can be seen clearly from the effective potential that when the angular momentum vanishes, $L = 0$, there is no turning point, and then the probe might be captured by the source.

As mentioned above, the source D3-branes are distributed on a disc in a plane defined by $y_1 = y_2 = y_3 = y_4 = 0$. So it would be interesting to study the motion of the probe in this plane or orthogonal to this plane. Let's first discuss the case for the probe moving in the disc plane.

(i-a).*In the disc plane.* In this case, one has $r^2 = \rho^2$, and

$$H_0 = 1 + \frac{R^4}{(\rho^2 - l^2)^2}, \tag{14}$$

where $\rho^2 = y_5^2 + y_6^2$. Indeed, the harmonic function is singular at the edge of the disc $\rho = l$. In this case, the effective potential becomes

$$V(\rho) = E\left[1 - \frac{1 + \frac{E}{2m_p}\left(1 + \frac{R^4}{(\rho^2-l^2)^2}\right)}{\left[1 + \frac{E}{m_p}\left(1 + \frac{R^4}{(\rho^2-l^2)^2}\right)\right]^2}\right] + \frac{L^2}{2m_p\rho^2}\frac{1}{\left[1 + \frac{E}{m_p}\left(1 + \frac{R^4}{(\rho^2-l^2)^2}\right)\right]^2}. \tag{15}$$

We first consider the far region, that is, $\rho^2 - l^2 >> R^2$. The potential has the following behavior:

$$\begin{aligned} V(\rho) \approx \quad & E\left[1 - \frac{m_p(E + 2m_p)}{2(E + m_p)^2} + \frac{m_p E(E + 3m_p)}{2(E + m_p)^3}\frac{R^4}{(\rho^2 - l^2)^2}\right] \\ & + \frac{m_p L^2}{2(E + m_p)^2\rho^2}\left[1 - \frac{2E^2}{m_p(E + m_p)}\frac{R^4}{(\rho^2 - l^2)^2}\right]. \end{aligned} \tag{16}$$

The force exerted on the probe mainly comes from the repulsive centrifugal force due to the non-zero angular momentum L. The effect of the source is of the sub-leading order. Compared to the single-center case, where $l = 0$, the sub-leading effect is enhanced due to the distribution of the source D3-branes. In the region near the edge of the disc, namely, $\rho^2 - l^2 << R^2$, the constant 1 in the harmonic function (14) can be dropped out. The effective potential reduces to

$$V(\rho) = E\left[1 - \frac{1 + \frac{\rho_*^4}{2(\rho^2-l^2)^2}}{\left(1 + \frac{\rho_*^4}{(\rho^2-l^2)^2}\right)^2}\right] + \frac{L^2}{2m_p\rho^2}\frac{1}{\left(1 + \frac{\rho_*^4}{(\rho^2-l^2)^2}\right)^2}, \tag{17}$$

where we introduced a characteristic scale $\rho_*^4 = ER^4/m_p$. At the distance $\rho^2 - l^2 >> \rho_*^2$, the effective potential takes the following form:

$$V(\rho) \approx \frac{3E\rho_*^4}{2(\rho^2 - l^2)^2} + \frac{L^2}{2m_p\rho^2}\left(1 - \frac{2\rho_*^4}{(\rho^2 - l^2)^2}\right). \tag{18}$$

In this region, the potential is a sum of two repulsive potentials. The first term is enhanced due to the distribution of the source, while the usually repulsive centrifugal potential (second term) is suppressed. When $\rho^2 - l^2 << \rho_*^2$,

$$V(\rho) \approx E - \frac{E(\rho^2 - l^2)^2}{2\rho_*^4} + \frac{L^2}{2m_p\rho^2}\frac{(\rho^2 - l^2)^4}{\rho_*^8}. \tag{19}$$

The repulsive potential (second term) in fact is independent of the energy and is suppressed in this case; the usually centrifugal potential (third term) becomes attractive and it is also suppressed due to the distribution effect of the source. From (19) we see that the potential is a constant E at the edge of the disc and the central force, $F(\rho) = -dV(\rho)/d\rho$, on the probe vanishes there. When $L^2 = 0$, the turning point is at $\rho_c^2 = l^2$, therefore, the probe will be captured and it will locate at $\rho_c = l$; when $L^2 \neq 0$, the turning point ρ_c is

$$\rho_c = \frac{\rho_*^2/\rho_{**} + \sqrt{(\rho_*^2/\rho_{**})^2 + 4l^2}}{2}, \tag{20}$$

where $\rho_{**}^2 = L^2/m_pE$. Therefore we can conclude the motion of the probe in the disc plane as follows. When the probe is far from the disc, the effect due to the distribution of source is of the sub-leading order. For the radial motion in the disc plane, the probe will be captured at the edge of the disc if its angular momentum vanishes; the probe will be bounced back and never be absorbed by the source if its angular momentum does not vanish.

(i-b). *In the hyperplane orthogonal to the disc.* When the probe is restricted in a hyperplane defined as $y_5 = y_6 = 0$, which is orthogonal to the disc plane discussed above, we have $\rho = 0$, and

$$H_0 = 1 + \frac{R^4}{r^2(r^2 + l^2)}, \tag{21}$$

where $r^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2$. In this case, $r = 0$ is a singular hyperplane for the harmonic function. In the asymptotically far region, we have the effective potential

$$\begin{aligned} V(r) \approx \quad & E\left[1 - \frac{m_p(E + 2m_p)}{2(E + m_p)^2} + \frac{m_pE(E + 3m_p)}{2(E + m_p)^3}\frac{R^4}{r^2(r^2 + l^2)}\right] \\ & + \frac{m_pL^2}{2(E + m_p)^2r^2}\left[1 - \frac{2E^2}{m_p(E + m_p)}\frac{R^4}{r^2(r^2 + l^2)}\right]. \end{aligned} \tag{22}$$

Once again, as expected, the distribution effect of the source is in the sub-leading order and the main contribution to the force on the probe comes from the repulsive centrifugal force. Unlike the case in the disc plane, however, the sub-leading effect is suppressed here. In the region near the disc, the effective potential has the following form:

$$V(r) = E\left[1 - \frac{1 + \frac{\rho_*^4}{2r^2(r^2+l^2)}}{\left(1 + \frac{\rho_*^4}{r^2(r^2+l^2)}\right)^2}\right] + \frac{L^2}{2m_pr^2}\frac{1}{\left(1 + \frac{\rho_*^4}{r^2(r^2+l^2)}\right)^2}. \tag{23}$$

At the distance $r >> \rho_*$, the effective potential takes the following form:

$$V(r) \approx \frac{3E\rho_*^4}{2r^2(r^2+l^2)} + \frac{L^2}{2m_p r^2}\left(1 - \frac{2\rho_*^4}{r^2(r^2+l^2)}\right). \tag{24}$$

Here the first repulsive potential term is suppressed, while the centrifugal repulsive potential is enhanced. When $r << \rho_*$, the effective potential reduces to

$$V(r) \approx E - \frac{Er^2(r^2+l^2)}{2\rho_*^4} + \frac{L^2}{2m_p}\frac{r^2(r^2+l^2)^2}{\rho_*^8}. \tag{25}$$

In this case, once again the usually centrifugal repulsive potential becomes attractive and it is enhanced, compared to the single-center case. The second term in (25) is a repulsive potential, which is also enhanced due to the distribution effect of the source. From (25) we see that the potential is a constant E at $r = 0$ and the central force on the probe vanishes there. Therefore the probe will be absorbed at $r = 0$ if the probe has no angular momentum. Otherwise, the probe will be bounced back at the turning point:

$$r_c^2 = \frac{\rho_*^4}{\rho_{**}^2} - l^2. \tag{26}$$

(ii). *In the non-extremal background.* In this case, the background represents the rotating D3-brane solution. Substituting (1) and (3) into (7) yields

$$S = -T_3 V_3 \int d\tau f^{-1}\left[\sqrt{h - f\omega^2} - 1 + f_0 - f + \frac{(1-f)l\sin^2\theta}{\sinh\alpha}\dot{\phi}\right], \tag{27}$$

where

$$\omega^2 = \frac{\dot{r}^2}{\tilde{h}} + r^2(\triangle\dot{\theta}^2 + \tilde{\triangle}\sin^2\theta\dot{\phi}^2 + \cos^2\theta\dot{\Omega}_3^2) - \frac{4ml\cosh\alpha}{r^4\triangle f}\sin^2\theta\dot{\phi}. \tag{28}$$

and

$$f_0 = 1 + \frac{R^4}{r^4\triangle}. \tag{29}$$

In this case, the static interaction potential will no longer vanish and can be written down as:

$$V_0(r,\theta) = m_p f^{-1}(\sqrt{h} - 1 + f_0 - f). \tag{30}$$

The potential depends on not only the radial position of the probe, but also the azimuth θ. Quite interesting is that when $\theta = \pi/2$, which means that the probe is in the disc plane discussed above, the effect due to the rotation of the source disappears. The form of potential implies that the static probe suffers from an attractive force from the source. The general motion of the probe is quite complicated. Here we consider the motion of probe in two self-consistent cases: One is $\theta = 0$ and $\Omega_3 = const.$, and the other is $\theta = \pi/2$ and $\Omega_3 = const.$. In the first case, which implies that the probe is moving in the hyperplane orthogonal to the disc plane, the action reduces to

$$S = -m_p \int d\tau f^{-1}[\sqrt{h - f\dot{r}^2/\tilde{h}} - 1 + f_0 - f], \tag{31}$$

and

$$f = 1 + \frac{\tilde{R}^4}{r^4 \triangle}, \quad \triangle = \tilde{\triangle} = 1 + \frac{l^2}{r^2}, \quad h = 1 - \frac{2m}{r^4 \triangle} = \tilde{h} = \frac{1}{\triangle}\left(1 + \frac{l^2}{r^2} - \frac{2m}{r^4}\right), \tag{32}$$

where $\tilde{R}^4 = \sqrt{R^8 + m^2} - m$. The dependence of motion on the angular coordinate ϕ disappears automatically, the radial effective potential can be expressed as

$$V(r) = E\left[1 - \frac{m_p h\tilde{h}}{2Ef}\left(1 - \frac{h}{(1 + f - f_0 + Ef/m_p)^2}\right)\right]. \tag{33}$$

We are interested in the so-called field theory limit, in which one has $f \approx f_0 = R^4/r^4\triangle$. And then the effective potential reduces to

$$V(r) = E\left[1 - \frac{r^4 \triangle h\tilde{h}}{2r_*^4}\left(1 - \frac{h}{(1 + r_*^4/r^4\triangle)^2}\right)\right]. \tag{34}$$

Here $r_*^4 = ER^4/m_p$. For the rotating D3-brane solution (1), there is a horizon r_+ determined by $\tilde{h} = 0$,

$$r_+^2 = \frac{1}{2}\left(\sqrt{l^4 + 8m} - l^2\right). \tag{35}$$

From (34) we find that the potential is a constant E at the horizon r_+ and the central force, $F(r) = -dV(r)/dr$, on the probe vanishes there. Furthermore, the turning point is just the horizon. Therefore the probe will be absorbed by the source and will locate at the horizon. Note that in this case the infinite red-shift hyperplane and event horizon of the background coincide with each other.

In the second case, namely, $\theta = \pi/2$ and $\Omega_3 = const.$, due to the dependence on the angular coordinate ϕ, the motion is still complicated. To simplify the problem, we further set $\dot{\phi} = 0$, namely, the angular momentum of the probe vanishes in this simplified situation. Thus the action of the probe is also expressed by (31), but

$$f = 1 + \frac{\tilde{R}^4}{r^4}, \quad f_0 = 1 + \frac{R^4}{r^4}, \quad h = 1 - \frac{2m}{r^4}, \quad \tilde{h} = 1 + \frac{l^2}{r^2} - \frac{2m}{r^4}. \tag{36}$$

In the field theory limit, the effective potential becomes

$$V(r) = E\left[1 - \frac{r^4 h\tilde{h}}{2r_*^4}\left(1 - \frac{h}{(1 + r_*^4/r^4)^2}\right)\right]. \tag{37}$$

In this case we have two turning points: one is the horizon r_+ and the other is $r_c^4 = 2m$, determined by $h = 0$. Note that the latter is just the infinite red-shift hyperplane in this case. At these two turning points the potential is the constant E, but the central force does not vanish there. Because of $r_c > r_+$ the probe will be bounced back at r_c. To compare this to the case of static source is interesting. For the static source case, one has

$h = \tilde{h} = 1 - 2m/r^4$. In that case, the turning point coincides with the horizon and the force on the probe vanishes. Thus the probe will be captured and will locate at the horizon.

In the calculations of the Wilsonian potential in the non-extremal supergravity solutions (for example see [29,30]), it has been assumed that the non-extremal branes "locate" at the horizon of non-extremal backgrounds. The analysis in this section provides an evidence of this assumption, at least for static background: indeed the brane can locate at the horizon of backgrounds. However, for the rotating background, the dynamics of the brane restricted in the disc plane poses a puzzle: the vanishing angular momentum brane cannot locate at the horizon. Perhaps we should consider the probe brane with the same angular velocity with the background. For such a brane, we expect that it can locate at the horizon.

III. THERMODYNAMICS OF A PROBE D3-BRANE IN THE ROTATING D3-BRANE BACKGROUND

Near the extremal limit, some thermodynamic quantities of the rotating D3-brane solution (1) can be expressed as follows [26]:

$$\begin{aligned}
&E = 3\pi^3 \kappa^{-2} m V_3, \\
&J = \pi^{7/4} \kappa^{-3/2} N^{1/2} m^{1/2} l V_3, \\
&\Omega = \pi^{5/4} \kappa^{-1/2} N^{-1/2} m^{-1/2} l r_+^2, \\
&T = 2^{-1} \pi^{1/4} \kappa^{-1/2} N^{-1/2} m^{-1/2} (2r_+^3 + l^2 r_+), \\
&S = 2\pi^{11/4} \kappa^{-3/2} N^{1/2} m^{1/2} r_+ V_3.
\end{aligned} \tag{38}$$

Here r_+ is given by (35); E denotes the energy above the extremality which equals the ADM mass of the black three-brane minus the mass of the corresponding extremal one; and J, Ω, T and S represent the angular momentum, angular velocity, Hawking temperature, and the entropy, respectively. In addition, $2\kappa^2 = 16\pi G = (2\pi)^7 g_s^2 \alpha'^4$ and G is the Newton gravitational constant in ten dimensions. These quantities satisfy the first law of thermodynamics

$$dE = TdS + \Omega dJ. \tag{39}$$

In [26] it has been observed that the heat capacity at a constant angular velocity, $C_\Omega = T(\partial S/\partial T)_\Omega$, diverges at $l^2 = 2r_+^2$, namely, $l^4/m = 8/3$, and beyond which the heat capacity becomes negative. This means that the $\mathcal{N}$=4 large N SYM corresponding to the rotating D3-brane configuration has a phase transition. Its critical behavior at the critical point has been investigated and some relevant critical exponents have been obtained in [26]. Recall that the SYM theory corresponding to this rotating D3-brane solution is in the Higgs phase. Thus the occurrence of phase transition in this rotating configuration is in agreement with the observation that phase transitions may be present in the Higgs phase [12].

Corresponding to this thermodynamic system, the Gibbs free energy, which is defined as $G = E - TS - \Omega J$, is

$$G_N = -\frac{1}{2}\pi^3 \kappa^{-2} r_+^2 (r_+^2 + l^2) V_3, \tag{40}$$

and the Helmholtz free energy, defined as $F = E - TS$, is

$$F_N = -\frac{1}{2}\pi^3\kappa^{-2}r_+^2(r_+^2 - l^2)V_3. \tag{41}$$

The Gibbs free energy does not change its sign, but Helmholtz free energy does at $r_+^2 = l^2$, that is, $l^4/m = 1$. However, it is not clear by now whether this change of sign is related or not to the Hawking-Page phase transition, which takes place in Schwarzschild-anti-de Sitter black holes. This phase transition has been interpreted by Witten as the confinement/unconfinement phase transition in the SYM theory [13].

Next we consider the thermodynamics of a probe D3-brane. For a static probe, as explained by Tseytlin and Yankielowicz [12], its distance to the source can be regarded as a mass scale in the SYM, and hence interaction static potential between the source and the probe can be interpreted as a contribution of massive states to the free energy of the large N SYM at the strong 't Hooft coupling. In the rotating D3-brane background (1), to keep relative static position of the probe to the rotating source, we have to let the probe rotate along an orbit in the direction ϕ with the same angular velocity $\dot{\phi} = \Omega$. The Euclidean action of such a probe is

$$\begin{aligned} I \equiv \beta G_p = T_3V_3\beta f^{-1} &\left(\sqrt{h - fr^2\tilde{\triangle}\Omega^2\sin^2\theta + \frac{4ml\Omega\cosh\alpha}{r^4\triangle}\sin^2\theta}\right. \\ &\left. - 1 + f_0 - f + \frac{(1-f)l\Omega}{\sinh\alpha}\sin^2\theta\right), \end{aligned} \tag{42}$$

where $\beta = 1/T$ is the inverse Hawking temperature. Since we are interested in thermodynamics of SYM theory through the Maldacena conjecture, we consider the field theory limit:

$$r \to u\alpha', \quad m \to m\alpha'^4, \quad l \to l\alpha', \quad \alpha' \to 0, \tag{43}$$

but g_s gets fixed. In this limit, we have $r_+^2 \to r_+^2\alpha'^2$ and all quantities in (38) keep same forms, only difference is that $2\kappa^2$ is replaced by $2\tilde{\kappa}^2 = (2\pi)^7 g_s^2$. Furthermore, equations (39), (40) and (41) remain same forms as well.

Considering the field theory limit (43), we have

$$f \approx f_0 = \frac{R^4}{\alpha'^4\triangle u^4}, \quad \triangle = 1 + \frac{l^2\cos^2\theta}{u^2}, \quad \tilde{\triangle} = 1 + \frac{l^2}{u^2}, \quad h = 1 - \frac{2m}{u^4\triangle}. \tag{44}$$

And the free energy of the probe reduces to

$$\begin{aligned} G_p = \frac{T_3V_3\alpha'^4\triangle u^4}{R^4} &\left(\sqrt{1 - \frac{2m}{u^4\triangle} - \frac{R^4\tilde{\triangle}\Omega^2}{\alpha'^2\triangle u^2}\sin^2\theta + \frac{2\sqrt{2ml}\Omega R^2}{\alpha'\triangle u^4}\sin^2\theta}\right. \\ &\left. - 1 - \frac{\sqrt{2ml}\Omega R^2}{\alpha'\triangle u^4}\sin^2\theta\right). \end{aligned} \tag{45}$$

Having considered α' appearing in the tension T_3 and R^4, the parameter α' disappears in the above free energy, which makes the contribution of the massive states to the free energy of the SYM theory be finite,

$$G_p = \frac{V_3 N \triangle u^4}{2\pi^2\lambda^2}\left(\sqrt{1-\frac{2m}{u^4\triangle}-\frac{\lambda\tilde{\triangle}\Omega^2}{\triangle u^2}\sin^2\theta+\frac{2\sqrt{2m\lambda}l\Omega}{\triangle u^4}\sin^2\theta}-1-\frac{\sqrt{2m\lambda}l\Omega}{\triangle u^4}\sin^2\theta\right), \tag{46}$$

where $\lambda = 2Ng_{\rm YM}^2 = 4\pi N g_s$ is the 't Hooft coupling constant. Unlike the case of the probe in the static non-extremal D3-brane background, there the free energy of the probe depends on the temperature of source and the mass scale u of scalar fields (see [12]), here the free energy depends not only on the temperature and mass scale, but also on the angular velocity of the source and the azimuth θ. In fact, the dependence on the azimuth θ means that the probe is separated by not only a single direction from the source D3-branes. In principle, we can eliminate m and l from (46) by using the expressions of the Hawking temperature and angular velocity in (38). In practice, however, the expression of the free energy would be quite complicated if we did. So we keep the form (46) in the following discussions.

From the expression (46) of the free energy we see that there is a minimum, which occurs when the square root in (46) is zero, namely,

$$1-\frac{2m}{u^4\triangle}-\frac{\lambda\tilde{\triangle}\Omega^2}{\triangle u^2}\sin^2\theta+\frac{2\sqrt{2m\lambda}l\Omega}{\triangle u^4}\sin^2\theta=0. \tag{47}$$

Having been given the free energy of the probe, we can acquire other thermodynamic quantities of the probe immediately. For example, its entropy, angular momentum, and heat capacity can be obtained by using following formulas:

$$S_p = -\left(\frac{\partial G_p}{\partial T}\right)_\Omega, \quad J_p = -\left(\frac{\partial G_p}{\partial \Omega}\right)_T, \quad C_{\Omega p} = T\left(\frac{\partial S_p}{\partial T}\right)_\Omega. \tag{48}$$

Below we discuss two special positions of the probe. One is $\theta = 0$. That is, the probe is in a hyperplane perpendicular to the disc plane of the source. The free energy (46) takes a very simple form in this case:

$$G_\perp = \frac{V_3 N u^2(u^2+l^2)}{2\pi^2\lambda^2}\left(\sqrt{1-\frac{2m}{u^2(u^2+l^2)}}-1\right). \tag{49}$$

The entropy of the probe is

$$S_p = \frac{2V_3N\sqrt{2m\lambda}l^4}{\pi\lambda^2(2r_+^4+3l^2r_+^2+l^4)}\frac{l^2+2r_+^2}{l^2-2r_+^2}\frac{1}{\triangle_*}\left[\frac{m(l^2-2r_+^2)}{l^4}-\frac{u^2(u^2+l^2)-m}{u^2+l^2}-u^2\triangle_*\right], \tag{50}$$

and its angular momentum

$$J_p = \frac{V_3N}{\pi^2\lambda^2}\frac{\sqrt{2m\lambda}(2r_+^2+l^2)}{(r_+^2+l^2)(l^2-2r_+^2)}\frac{l}{\triangle_*}\left[\frac{m(l^2-2r_+^2)}{r_+^2(2r_+^2+3l^2)}+\frac{u^2(u^2+l^2)-m}{u^2+l^2}-u^2\triangle_*\right] \tag{51}$$

where $\triangle_* = \sqrt{1-2m/u^2(u^2+l^2)}$. We can also calculate the heat capacity of the probe, but because of the complexity of its expression, we are not going to present it here. We find that the entropy and angular momentum of the probe diverge at $l^2 = 2r_+^2$ and $\triangle_* = 0$, the heat capacity has also the same property. The divergence of thermodynamic quantities means that the existence of critical points and the occurrence of phase transitions. The first point is just the thermodynamically stable boundary of the source rotating D3-branes [26], at which the ratio of the critical angular velocity to the critical temperature is

$$\gamma_s \equiv \left.\frac{\Omega}{T}\right|_s = \left.\frac{2\pi l r_+}{2r_+^2 + l^2}\right|_{l^2=2r_+^2} = \frac{\pi}{\sqrt{2}}. \tag{52}$$

At the second point, we can also get a similar relation from $\triangle_* = 0$,

$$\gamma_p \equiv \left.\frac{\Omega}{T}\right|_p = \left.\frac{2\pi l r_+}{2r_+^2 + l^2}\right|_{2m=u^2(u^2+l^2)} = \frac{\sqrt{2}\pi l\sqrt{\sqrt{l^4+4u^2(u^2+l^2)}-l^2}}{\sqrt{l^4+4u^2(u^2+l^2)}}, \tag{53}$$

which depends on the value u/l. For example, when $u/l = 1$, one has $\gamma_p = 2\pi/3$. The second critical point is just the extension of the so-called maximal temperature observed in [12] when the probe is in the static D3-brane background. Furthermore, $\triangle_* = 0$ means that the probe is just at the horizon of the rotating source. In the static background, the first critical point is absent. In the rotating D3-brane background, for fixed angular velocity and mass scale, if $\gamma_s < \gamma_p$, the first critical point will not occur, that is, the whole system is already unstable due to the probe before the occurrence of the thermodynamic instability of the source; if $\gamma_s > \gamma_p$, the second critical point is then absent because the whole system is already unstable due to the thermodynamic instability of source before reaching the second critical temperature.

In [12] it has been observed that at the low-temperature or long-distance ($u \to \infty$) limit, the free energy of the probe can be expressed as

$$G_p^\infty = G_{N+1} - G_N - G_1, \tag{54}$$

where G_{N+1}, G_N, and G_1 denotes the free energies for the $N+1$ coinciding D3-branes, N coinciding D3-branes and one D3-brane, respectively. Hence the free energy of the probe can be explained very well by the contribution of the massive states to the free energy of SYM theory at strong 't Hooft coupling. In our case, taking the limit $u \to \infty$, from (49) we have

$$G_\perp^\infty = -\frac{V_3 N r_+^2 (r_+^2 + l^2)}{4\pi^2\lambda^2}, \tag{55}$$

while from (40)

$$G_N = -\frac{V_3 N^2 r_+^2 (r_+^2 + l^2)}{8\pi^2\lambda^2}. \tag{56}$$

As expected, the relation (54) holds as well in this case.

When $\theta = \pi/2$, the free energy of the probe is

$$G_\parallel = \frac{V_3 N u^4}{2\pi^2\lambda^2}\left(\sqrt{1 - \frac{2m}{u^4} - \frac{\lambda\Omega^2(u^2+l^2)}{u^4} + \frac{2\sqrt{2m\lambda}l\Omega}{u^4}} - 1 - \frac{\sqrt{2m\lambda}l\Omega}{u^4}\right). \tag{57}$$

Its entropy is

$$S_p = \frac{V_3 N\sqrt{2m\lambda} l^3}{\pi\lambda^2(2r_+^4+3l^2r_+^2+l^4)}\frac{l^2+2r_+^2}{l^2-2r_+^2}\frac{1}{\triangle_{**}}\left[\left(2m+\sqrt{2m\lambda}l\Omega\triangle_{**}-\sqrt{2m\lambda}l\Omega\right)\frac{l^2-2r_+^2}{l^3}\right.$$
$$\left. - \sqrt{2m\lambda}\Omega+\lambda l\Omega^2+\sqrt{2m\lambda}\Omega\triangle_{**}\right], \qquad (58)$$

and the angular momentum

$$J_p = \frac{V_3 N}{2\pi^2\lambda^2}\frac{\sqrt{2m\lambda}(2r_+^2+3l^2)}{(r_+^2+l^2)(l^2-2r_+^2)}\frac{1}{\triangle_{**}}\left[\left(2m+\sqrt{2m\lambda}l\Omega\triangle_{**}-\sqrt{2m\lambda}l\Omega\right)\frac{l(l^2-2r_+^2)}{r_+^2(2r_+^2+3l^2)}\right.$$
$$\left. + \sqrt{2m\lambda}\Omega-\lambda l\Omega^2-\sqrt{2m\lambda}\Omega\triangle_{**}\right], \qquad (59)$$

where

$$\triangle_{**} = \sqrt{1-\frac{2m}{u^4}-\frac{\lambda\Omega^2(u^2+l^2)}{u^4}+\frac{2\sqrt{2m\lambda}l\Omega}{u^4}}.$$

The same happens as the first case. That is, the entropy and angular momentum of the probe is divergent at the source instability point $l^2 = 2r_+^2$ and $\triangle_{**} = 0$. The remarks about the thermodynamic instability in the first case is applicable here as well. However, from (57) we see that in the long-distance limit (large u)

$$G_{\|} \approx -\frac{V_3 N}{4\pi^2\lambda^2}\left[r_+^2(r_+^2+l^2)+\lambda\Omega^2(u^2+l^2)\right]. \qquad (60)$$

That is, when $u \to \infty$, the free energy is divergent. Obviously the relation (54) does not hold in this case. We have not yet understood this phenomenon on the SYM side. For a general θ, the thermodynamic properties of the probe is similar and the second critical point is just determined by the equation (47). In addition, it is worth noting that when $\Omega = 0$ in (46), namely, for a static probe, the relation (54) can be recovered in the limit $u \to \infty$ for any azimuth θ. In this case, the second critical point is $2m = u^2(u^2 + l^2\cos^2\theta)$, which corresponds to that the probe is just at the infinite red-shift hyperplane of the background.

To summarize, we investigated the dynamics and thermodynamics of a probe D3-brane in the rotating D3-brane background. As is well-known, the extremal limit of the rotating D3-brane configuration is a multicenter solution of D3-branes, which are distributed uniformly on a disc; the excitations of rotating D3-branes are thermodynamically stable up to a critical angular momentum density, beyond which the heat capacity is negative. When we discussed the dynamics of the probe, we emphasized how the probe is captured by the source and what is the effect of the thermodynamic stability boundary on the motion of the probe. In the extremal background of the rotating D3-branes, the probe will be bounced back at some turning point if its angular momentum does not vanish. Otherwise, the probe will be absorbed by the source. When the probe is restricted in the disc plane, the probe will be captured and will locate at the edge of the disc; when the probe moves in the hyperplane which is orthogonal to the disc plane, the probe will be absorbed at the center of the disc. In the near-extremal background, when $\theta = 0$, which means that the probe is in the hyperplane orthogonal to the disc plane, the probe will be captured and locate at the horizon of the rotating D3-branes; In the disc plane, the probe will be bounced back at a turning point, which is just the infinite red-shift hyperplane of the background, even when the probe has zero angular momentum. We observed that a probe with vanishing angular momentum is always captured and will locate at the horizon if the background is the static D3-brane configuration. The dynamic analysis of the probe brane provides an evidence of the assumption: the non-extremal brane "locates" at the horizon of the supergravity background, at least for the static source. In addition, nothing special happens at the thermodynamically stable boundary of the rotating D3-branes from the point of view of the motion of the probe.

The free energy was obtained of a probe which keeps a relative static position to the rotating D3-branes, namely, the probe rotates along an orbit in the ϕ-direction with the same angular velocity $\dot{\phi} = \Omega$. The free energy depends on not only the Hawking temperature of the source and the distance to the source, but also the angular velocity and the azimuth θ. From the free energy some thermodynamic quantities, for example, the entropy, angular momentum, and heat capacity, of the probe can be worked out. We found that there are two critical points, at which those thermodynamic quantities of the probe diverge. One point is just the thermodynamic stable boundary of the rotating D3-branes; the other is related to the source parameters(Hawking temperature and angular velocity) and the position of the probe. The latter can be regarded as the mass scale of scalar fields in the SYM theory if one interpreted the free energy of the probe as the contribution of massive states to the one of SYM theory at the strong 't Hooft coupling. In addition, we found that except the case of $\theta = 0$, the free energy of the probe diverges in the long-distance limit. This is a new feature and does not appear in the static background. To further understand the thermodynamic behavior of the probe would be quite interesting on the SYM theory side.

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